ON THE THEORY OF MOTIONS OF THE PRANDTL-MEYER TYPE

(K TEORII DVIZHENII TIPA PRANDTLIA-MAIERA)

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Planar motion of the Prandtl-Meyer type is self-similar: by introducing a cylindrical system of coordinates (in which the z-axis coincides with the edge of the sharp angle at the surface of the body over which the gas flows) all quantities become functions of ϕ only. Flows in which the initial distribution of parameters in the stream moving toward the edge is non-uniform may be of interest; in particular, such flows may occur in the presence of a bow shock wave in front of a body whose intensity decreases with distance from the body.

If the initial distribution of parameters is considered to be known and to have almost constant values, the corresponding problem can be solved by the method of perturbations

$$v_r(\varphi) + v_r'(r, \varphi), \qquad v_\varphi(\varphi) + v_\varphi'(r, \varphi) \tag{1}$$

taking the solution of a self-similar problem [1] for the zeroth approximation

$$v_\varphi = c = c_* \cos n\varphi, \qquad p = p_* (\cos n\varphi)^\lambda \qquad \left(\lambda = \frac{1+n^2}{n^2}\right)$$

$$v_r = \frac{c_*}{n}\sin n\varphi, \qquad \rho = \rho_* (\cos n\varphi)^\mu \qquad \left(\mu = \frac{1-n^2}{n^2}\right) \qquad \left(n^2 = \frac{\gamma-1}{\gamma+1}\right) \tag{2}$$

Here γ is the ratio of specific heats, the asterisk denotes the critical value of the starred quantity, and the rest of the designations are well-known. For small values of v_r', v_ϕ', ρ' one can obtain the system of linear equations

$$v_r \frac{\partial \rho'}{\partial R} + v_\varphi \frac{\partial \rho'}{\partial \varphi} + \rho \frac{\partial v_r'}{\partial R} + \rho \frac{\partial v_\varphi'}{\partial \varphi} = \frac{d \ln \rho}{d\varphi} v_\varphi \rho - \frac{d\rho}{d\varphi} v_\varphi' - \rho v_r'$$

$$v_r \frac{\partial v_r'}{\partial R} + v_\varphi \frac{\partial v_r'}{\partial \varphi} + \frac{v_\varphi^2}{\rho} \frac{\partial \rho'}{\partial R} = v_\varphi v_\varphi' \qquad (R = \lg r) \tag{3}$$

$$v_r \frac{\partial v_\varphi'}{\partial R} + v_\varphi \frac{\partial v_\varphi'}{\partial \varphi} + \frac{v_\varphi^2}{\rho} \frac{\partial \rho'}{\partial \varphi} = \frac{d \ln \rho}{d\varphi} v_\varphi v_\varphi' + \frac{v_\varphi}{\rho}\left(v_\varphi \frac{d \ln \rho}{d\varphi} - 2\frac{dv_\varphi}{d\varphi}\right)\rho' - v_\varphi v_r'$$

Solving Equation (3) for derivatives with respect to R, the system obtained can be written in the form of the vector equation [2]

$$\frac{\partial \mathbf{x}}{\partial R} = A \frac{\partial \mathbf{x}}{\partial \varphi} + \Psi \qquad (x_1 = \rho', x_2 = v_r', x_3 = v_\varphi') \tag{4}$$

where $A = \| a_{ij}(\phi) \|$ is the matrix

$$\| a_{ij} \| = \begin{Vmatrix} \dfrac{-v_\varphi v_r}{v_r^2 - v_\varphi^2} & \dfrac{\rho v_\varphi}{v_r^2 - v_\varphi^2} & \dfrac{-\rho v_r}{v_r^2 - v_\varphi^2} \\ \dfrac{v_\varphi^3}{\rho (v_r^2 - v_\varphi^2)} & \dfrac{-v_r v_\varphi}{v_r^2 - v_\varphi^2} & \dfrac{v_\varphi^2}{v_r^2 - v_\varphi^2} \\ -\dfrac{v_\varphi^2}{\rho v_r} & 0 & -\dfrac{v_\varphi}{v_r} \end{Vmatrix} \tag{5}$$

We will introduce the vector $\mathbf{y}$ in place of $\mathbf{x}$ by means of the relations

$$\mathbf{x} = B\mathbf{y}, \qquad \mathbf{y} = B^{-1}\mathbf{x}$$

where $B = \| b_{ij}(\phi) \|$ is a matrix such that Det $|B| \neq 0$, and B^{-1} is the inverse matrix; we have

$$B \frac{\partial \mathbf{y}}{\partial R} = AB \frac{\partial \mathbf{y}}{\partial \varphi} + \Psi_1, \qquad \frac{\partial \mathbf{y}}{\partial R} = B^{-1}AB \frac{\partial \mathbf{y}}{\partial \varphi} + \Psi^2 \tag{6}$$

After finding the eigenvalues λ_k of the characteristic equation Det $|A - \lambda| = 0$

$$\lambda_1 = 0, \qquad \lambda_2 = -\frac{v_\varphi}{v_r}, \qquad \lambda_3 = -\frac{2 v_r v_\varphi}{v_r^2 - v_\varphi^2} \tag{7}$$

the elements b_{ik} can be determined from the system

$$\sum_{j=1}^{3} (a_{ij} - \delta_{ij} \lambda_k) b_{jk} = 0, \qquad \delta_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases} \qquad \begin{pmatrix} i = 1, 2, 3 \\ k = 1, 2, 3 \end{pmatrix} \tag{8}$$

such that the matrix $B^{-1}AB$ will have the diagonal form

$$\| b_{jk} \| = \begin{Vmatrix} -\dfrac{\rho}{v_\varphi} & 0 & \dfrac{\rho}{v_\varphi} \dfrac{v_r^2 + v_\varphi^2}{v_t^2 - v_\varphi^2} \\ 0 & -\dfrac{v_r}{v_\varphi} & -\dfrac{2 v_r v_\varphi}{v_r^2 - v_\varphi^2} \\ 1 & 1 & 1 \end{Vmatrix} \tag{9}$$

The system (6) can be reduced to the following form:

$$\frac{\partial y_1}{\partial R} = -y_1 - (1 - n^2) y_2 - \left(1 - \frac{2n^2}{1 - n^2 \operatorname{ctg}^2 n\varphi}\right) y_3$$

$$\frac{\partial y_2}{\partial R} + \frac{n}{\operatorname{tg} n\varphi}\frac{\partial y_2}{\partial \varphi} = n^2\left(1 - \frac{2}{1 + (n^2 - 1)\cos^2 n\varphi}\right) y_2 \tag{10}$$

$$\frac{\partial y_3}{\partial R} + \frac{2n \operatorname{tg} n\varphi}{\operatorname{tg}^2 n\varphi - n^2}\frac{\partial y_3}{\partial \varphi} = -\frac{(1 - n^2)\left[1 - (n^2 + 1)\cos^2 n\varphi\right]}{1 + (n^2 - 1)\cos^2 n\varphi} y_2 + \frac{2n^2}{1 - (n^2 + 1)\cos^2 n\varphi} y_3$$

For the solution of concrete problems, y_2, y_3 and y_1 are successively determined from the second, third and first of these equations.

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